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Variational model of bright beam evolution in weakly non-local media *

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Abstract.

In this work we model the nonlinear propagation of bright Gausian beams in the presence of a weak nonlocality. The variational results indicate an increase of the soliton power with increasing the nonlocality. Conditionsfor generating 'breathing' 1D solitons are derived.

1 Introduction

Generally, nonlocality means that the response of a material to an external action at a particular point depends also on the action on neighboring points. In a plasma the processes of heating and ionization are known to cause nonlocal response [1]. The long lifetime of optically pumped atoms allows the atomic diffusion to transport the excitation away from the interaction region. If the mean free path of the atoms is small compared to the diameter of the laser beam, the refractive index change extends beyond the laser beam cross-section [2]. Drift and/or diffusion of photoexcited carriers cause a nonlocal response in photorefractive materials too [3, 4]. Spatial nonlocality in the nonlinear response is present also in Bose-Einstein condensates when the particles exhibit long-range interactions [5, 6] or the localization of the condensate increases. The large nonlinearity of nematic liquid crystals stems from light-induced molecular reorientation. Due to elastic intermolecular forces the refractive-index change can extend well-beyond the excitation region [7]. It is apparent that heat conduction in materials with thermal nonlinearity results in nonlocal changes in their refractive indices [8]. Not that obvious, the analogy between parametric interaction in quadratic media and nonlocal cubic media [9, 10] has lead to the understanding that quadratic and nonlocal Kerr solitons are equivalent [11]. In this case the

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quadratic response depends on the square of the amplitude of the fundamental wave, not on the field intensity as expected for cubic response.

It is worth mentioning that while in most analyses the nonlocality in space is considered to be symmetric, the temporal response function can be asymmetric (as in the case of Raman effect on optical pulses [12]). The width of the response function relative to the width of the intensity profile determines the degree of the nonlocality [13]. In the local limit, the response becomes a δ-function. In the highly-nonlocal limit the beam evolution is described by the equation of a linear harmonic oscillator [14]. The special logarithmic nonlinearity allows exact analytical treatment [15]. In the case of weak nonlocality, exact analytical one-dimensional soliton solutions are found for both bright and dark solitons [16]. Important recent results show theoretically that two-dimensional ring vortex solitons should exist in self-focusing nonlinear media in the regime of strong nonlocality [17, 18].

2 Theoretical model

2.1 General formulation of the problem

The evolution of the slowly-varying electric field amplitude $E(x, y, z)$ is described by the two-dimensional generalized nonlinear Schrödinger equation which takes account of the beam diffraction, the self-phase modulation (SPM), and the weak anisotropic ($\gamma_x \neq \gamma_y$) nonlocality

$$i\partial_z E + \frac{1}{2k}(\partial_x^2 + \partial_y^2)E + k^{SPM}|E|^2 E + \gamma_x(\partial_x^2|E|^2)E + \gamma_y(\partial_y^2|E|^2)E = 0. \quad (1)$$

The (2+1)D Euler-Lagrange equation has the form

$$\frac{\partial}{\partial x}\frac{\partial \hat{L}}{\partial(\partial E^*/\partial x)} + \frac{\partial}{\partial y}\frac{\partial \hat{L}}{\partial(\partial E^*/\partial y)} + \frac{\partial}{\partial z}\frac{\partial \hat{L}}{\partial(\partial E^*/\partial z)} - \frac{\partial \hat{L}}{\partial E^*} = 0. \quad (2)$$

The density of the Lagrange operator $\hat{L}$ corresponding to Eq. 1 is

$$\hat{L} = ik(E\partial_z E^* - E^*\partial_z E) + \left|\partial_x E\right|^2 + \left|\partial_y E\right|^2 - k[k^{SPM}|E|^4 - \gamma_x(\partial_x|E|^2)^2 - \gamma_y(\partial_y|E|^2)^2]. \quad (3)$$

In essence [19], the variational approach requires selection of a trial function $E = E(x, y, q_i(z)), i = 1...n$ depending not only on x, y, but also on suitable variational variables (functions) $q_i(z)$. They have to be chosen by physical reasons. After substituting the trial function in $\hat{L}$ and integrating the result over the transverse coordinates x and y, one gets the Lagrangian $< \hat{L} >$

$$\langle \hat{L}(\dot{q}_i(z), q_i(z)) \rangle = \int\int \hat{L}(x, y, \dot{q}_i(z), q_i(z)dxdy \quad (4)$$

which has to satisfy the system of Lagrange equations

$$\frac{d}{dz}\frac{\partial\langle\hat{L}(\dot{q}_i(z),q_i(z))\rangle}{\partial\dot{q}_i(z)} - \frac{\partial\langle\hat{L}(\dot{q}_i(z),q_i(z))\rangle}{\partial q_i(z)} = 0. \qquad (5)$$

With dots we denote differentiation with respect to the propagation coordinate z. The result of the variational procedure is a system of ordinary differential equations (ODEs) for the variational parameters q_i [19, 20].

2.2 Two-dimensional results

In this work we analyze Gaussian beam described by

$$E(x,y,z) = A(z)\exp\{-x^2/\sigma_x^2(z) - y^2/\sigma_y^2(z) + i\Phi(x,y,z)\}, \qquad (6)$$

where $\Phi(x,y,z) = (k/2)\{\vartheta_x(z)x^2 + \vartheta_y(z)y^2\} + \varphi(z)$. Following the described variational procedure we derived the following system of ODEs for the variational variables

$$\begin{array}{l}
\dot{\sigma}_x = \vartheta_x\sigma_x \\
\dot{\vartheta}_x = -\vartheta_x^2 + 4/(k^2\sigma_x^4) - k^{SPM}A^2/(k\sigma_x^2) + 6\gamma_x A^2/(k\sigma_x^4) + 2\gamma_y A^2/(k\sigma_x^2\sigma_y^2) \\
\dot{\sigma}_y = \vartheta_y\sigma_y \\
\dot{\vartheta}_y = -\vartheta_y^2 + 4/(k^2\sigma_y^4) - k^{SPM}A^2/(k\sigma_y^2) + 6\gamma_y A^2/(k\sigma_y^4) + 2\gamma_x A^2/(k\sigma_x^2\sigma_y^2) \\
\dot{\varphi} = (-1/k)(\sigma_x^{-2} + \sigma_y^{-2}) + (3/4)k^{SPM}A^2 - 2(\gamma_x\sigma_x^{-2} + \gamma_y\sigma_y^{-2}) \\
(d/dz)(\pi k\sigma_x\sigma_y A^2) = 0.
\end{array} \qquad (7)$$

The last equation is nothing else but the energy conservation law $P = P_0 = (\pi/2)\sigma_{x0}\sigma_{y0}A_0^2 = (\pi/2)\sigma_x\sigma_y A^2$. Making use of this law, after routine transformations, one gets two second-order equations describing the evolution of the beam's widths

$$\begin{array}{l}
(\pi k\sigma_x^2\sigma_y)\ddot{\sigma}_x = 4\pi\sigma_y/(k\sigma_x) - 2k^{SPM}P_0 + 12\gamma_x P_0/\sigma_x^2 + 4\gamma_y P_0/\sigma_y^2 \\
(\pi k\sigma_y^2\sigma_x)\ddot{\sigma}_y = 4\pi\sigma_x/(k\sigma_y) - 2k^{SPM}P_0 + 12\gamma_y P_0/\sigma_y^2 + 4\gamma_x P_0/\sigma_x^2.
\end{array} \qquad (8)$$

2.3 Reduction to one transverse dimension

Under certain conditions (e.g. planar waveguides, highly elliptical laser beams) the dimensionality of the generalized Schrödinger equation can be reduced. Considering nonlinear propagation of an 1D Gaussian beam ($E(x,y,z) = A(z)exp[-x^2/\sigma_x^2(z) + ik\vartheta_x(z)x^2/2 + i\varphi(z)]$) and following the described procedure, one gets the following system of equations for the evolution of the non-trivial variational variables

$$\begin{array}{l}
\dot{\sigma}_x = \vartheta_x\sigma_x \\
\dot{\vartheta}_x = -\vartheta_x^2 + 4/(k^2\sigma_x^4) - \sqrt{2}k^{SPM}A^2/(k\sigma_x^2) + 6\sqrt{2}\gamma_x A^2/(k\sigma_x^4) \\
\dot{\varphi} = -1/(k\sigma_x^2) + (3/4)\sqrt{2}k^{SPM}A^2 - 2\sqrt{2}\gamma_x A^2/\sigma_x^2 \\
(d/dz)(\sqrt{2\pi}k\sigma_x\sigma_y A^2) = 0.
\end{array} \qquad (9)$$

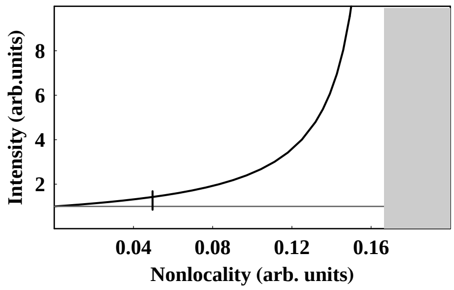

Figure 1. Rise of the soliton intensity with increasing the dimensionless nonlocality γ_x/σ_{xs}^2. Shaded region - nonlinear diffraction overcompensating the spatial self-phase modulation. The vertical bar denotes the nonlocality used in generating Fig. 3.

Taking the 1D energy conservation in to account ($P = P_0 = \sqrt{\pi/2}\sigma_{x0}A_0^2 = \sqrt{\pi/2}\sigma_x A^2$) we arrive at the second-order ODE governing the evolution of the beam width σ_x

$$\ddot{\sigma}_x = 4/(k^2\sigma_x^3) - 2k^{SPM}P_0/(\sqrt{\pi}k\sigma_x^2) + 12\gamma_x P_0/(\sqrt{\pi}k\sigma_x^4). \quad (10)$$

This equation admits a soliton solution σ_{xs} when $\sigma_x(z) = \sigma_x(z=0) \equiv \sigma_{x0} = \sigma_{xs}$ and $\dot{\sigma}_x \equiv 0$ (i.e. when the input beam remains a plane wave; $\vartheta_x \equiv \vartheta_{x0} = 0$). The soliton power P_x^{sol} and intensity I_x^{sol} in the 1D nonlocal case analyzed here become

$$P_x^{sol} = \sqrt{\frac{\pi}{2}}\sigma_{xs}\frac{2\sqrt{2}}{kk^{SPM}\sigma_{xs}^2 - 6k\gamma_x}; I_x^{sol} = P_x^{sol}/(\sigma_{xs}\sqrt{\pi/2}). \quad (11)$$

It is quite apparent that in a nonlocal medium the soliton radius σ_{xs} has to exceed a minimal radius $\sigma_{xs,min} = (6\gamma_x/k^{SPM})^{1/2}$. In the local approximation $\gamma_x = 0$ and $\sigma_{xs,min} = 0$, which is unphysical. In the general model equation (Eq. 1) the last two terms appear as describing an intensity-dependent diffraction. This motivates the physical interpretation of the obtained results showing that the higher the medium nonlocality, the larger the minimal achievable 1D beam width. The monotonic increase of the power/intensity needed for the beam's self-channeling is depicted in Fig. 1. Fig. 2 is aimed to show how well the obtained variational results match the exact ones obtained by Bang et al. [16]. Qualitatively, the Gaussian trial function seems to approximate reasonably well the shape of the nonlocal soliton and the deviation decreases for higher nonlocality. If the incoming beam is focused, the beam' wavefront curvature $\vartheta_{x0} \neq 0$. Integrating Eq. 10 ones we obtained

$$\dot{\sigma}_x = \frac{8\gamma_x P_0}{\sqrt{\pi}k}(\sigma_{x0}^{-3} - \sigma_x^{-3}) + \frac{4}{k^2}(\sigma_{x0}^{-2} - \sigma_x^{-2}) - \frac{4k^{SPM}P_0}{\sqrt{\pi}k}(\sigma_{x0}^{-1} - \sigma_x^{-1}) + (\vartheta_{x0}\sigma_{x0})^2. \quad (12)$$

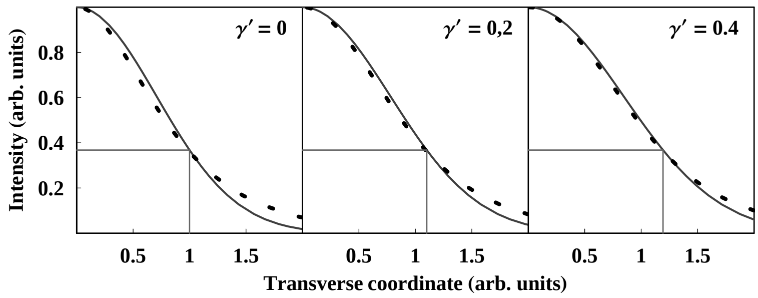

Figure 2. Intensity profiles of 1D nonlocal spatial solitons for different values of the parameter $\gamma\prime = 2k\gamma_x I_x^{sol}$. Solid curve - variational results, dotted - exact analytical results (see [16]).

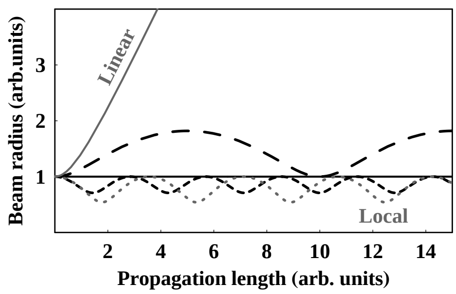

Figure 3. Beam radius vs. nonlinear propagation path length for $\gamma_x/\sigma_{xs}^2 = 0.05$ and $P/P_x^{sol} = 2/3, 3/2$, and 1 (short-dashed curve, long-dashed curve, and horizontal solid line, respectively). The beam radii in the linear and in the local nonlinear regimes (dotted curve) are shown for comparison.

In Fig. 3 we show the evolution of the beam radius along the nonlinear medium in the characteristic cases of nonlocal soliton formation (solid line), generation of ’breathing’ solitons (short and long-hashed curves) and compare them with beam radii in a linear and purely local nonlinear regime. In the case of a non-stationary propagation along the NLM the extrema $\sigma_{x,extr}$ of the beam width $\sigma_x(z)$ has to satisfy the condition $\dot{\sigma}_x = 0$, i.e.

$$\left(1 - \frac{kk^{SPM}P_0\sigma_{x0}}{\sqrt{\pi}} + \frac{2k\gamma_x P_0}{\sqrt{\pi}\sigma_{x0}} + \frac{k^2\vartheta_{x0}^2\sigma_{x0}^4}{4}\right)\left(\frac{\sigma_{x,extr}}{\sigma_{x0}}\right)^3 + \frac{kk^{SPM}P_0\sigma_{x0}}{\sqrt{\pi}}\left(\frac{\sigma_{x,extr}}{\sigma_{x0}}\right)^2 = \frac{\sigma_{x,extr}}{\sigma_{x0}} + \frac{2k\gamma_x P_0}{\sqrt{\pi}\sigma_{x0}}. \quad (13)$$

The above equation has two real and positive solutions corresponding to minimal and maximal physical beam radii $\sigma_{x,min}$ and $\sigma_{x,max}$, respectively, only if

$$1 - kk^{SPM}P_0\sigma_{x0}/\sqrt{\pi} + 2k\gamma_x P_0/(\sqrt{\pi}\sigma_{x0}) + k^2\vartheta_{x0}^2\sigma_{x0}^4/4 < 0. \quad (14)$$

Since $\sigma_x(z)$ is allowed to oscillate between $\sigma_{x,min}$ and $\sigma_{x,max}$, the inequality given by Eq. 14 is the condition to create a 'breathing' soliton.

3 Conclusions

The comparison of our variational result with the exact analytical solution for an 1D bright spatial soliton [16] allows to state that the variational method is a powerful method which is well suited to describe nonlinear beam evolution in nonlocal media. Analytical results for the increase of the soliton power with increasing the nonlocality and for generating 'breathing' 1D solitons are derived and discussed.

Acknowledgments

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